Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 2, May 2003, pp. 139–152.
Printed in India

The Jacobian of a nonorientable Klein surface

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MS received 10 July 2002; revised 17 January 2003

Abstract. Using divisors, an analog of the Jacobian for a compact connected nonorientable Klein surface Y is constructed. The Jacobian is identified with the dual of the space of all harmonic real one-forms on Y quotiented by the torsion-free part of the first integral homology of Y. Denote by X the double cover of Y given by orientation. The Jacobian of Y is identified with the space of all degree zero holomorphic line bundles L over X with the property that L is isomorphic to $\sigma^*\overline{L}$, where σ is the involution of X.

Keywords. Nonorientable surface; divisor; Jacobian

1. Introduction

Let Y be a compact connected nonorientable Riemann surface, that is, each transition function is either holomorphic or anti-holomorphic. We consider surfaces without boundary. Let X denote the double cover of Y given by the local orientations. So X is a compact connected Riemann surface.

In §2., we define a morphism from Y to $\overline{\mathbb{H}}$, the closure of the upper half-plane in the Riemann sphere $\widehat{\mathbb{C}}$. Let $\mathrm{Div}_0(Y)$ denote the group defined by all formal finite sums of the form $\sum n_i y_i$, where $n_i \in \mathbb{Z}$ with $\sum n_i = 0$ and $y_i \in Y$. We call such a divisor D to be principal if there is a morphism (see §2. for the definition of morphism) u from Y to $\overline{\mathbb{H}}$ with the property that

$$D = u^{-1}(0) - u^{-1}(\infty).$$

Let $J_0(Y)$ denote the quotient of $\mathrm{Div}_0(Y)$ by its subgroup consisting of all principal divisors. This $J_0(Y)$ is the analog of the Jacobian for a nonorientable Riemann surface.

Harmonic one-forms are defined on Y. Let $H^1_{\mathbb{R}}(Y)$ denote the space of all harmonic real one-forms on Y. The torsion-free part of $H_1(Y,\mathbb{Z})$ is a subgroup of $\mathscr{H}^1_{\mathbb{R}}(Y)^*$. The quotient is identified with $J_0(Y)$. This is proved by showing that $\mathscr{H}^1_{\mathbb{R}}(Y)$ is identified with the space of all holomorphic one-forms ω on X satisfying the identity $\overline{\omega} = \sigma^*\omega$, where σ is the nontrivial automorphism of the double cover X of Y (Theorem 2.7).

For a holomorphic line bundle L over X, the pullback $\sigma^*\overline{L}$ is again a holomorphic line bundle over X. We show that $J_0(Y)$ is identified with the group of all holomorphic line bundles L over X for which the holomorphic line bundle $\sigma^*\overline{L}$ is isomorphic to L (Theorem 4.2).

A compact Riemann surface is a smooth projective curve over $\mathbb{C}$. Conversely, every smooth projective curve over $\mathbb{C}$ corresponds to a compact Riemann surface. If we take a

smooth projective curve $X_{\mathbb{R}}$ defined over $\mathbb{R}$, then using the inclusion of $\mathbb{R}$ in $\mathbb{C}$ we get a smooth projective curve $X_{\mathbb{C}}$ over $\mathbb{C}$. Now, since the involution of $\mathbb{C}$ defined by conjugation fixes $\mathbb{R}$, the complex curve $X_{\mathbb{C}}$ is equipped with an anti-holomorphic involution that reverses the orientation. Conversely, every complex projective curve equipped with an anti-holomorphic involution is actually defined over $\mathbb{R}$. If the involution does not have any fixed points, that is, the curve does not have any real points, then it is called an imaginary curve.

Therefore, a nonorientable Riemann surface Y (without boundary) corresponds to an imaginary algebraic curve defined over $\mathbb{R}$. The Jacobian of the complexification $Y\mathbb{C}$ is also the complexification of a variety defined over $\mathbb{R}$. The Jacobian $J_0(Y)$ coincides with this variety defined over $\mathbb{R}$.

2. Divisors on a nonorientable surface

Let Y be a compact connected nonorientable surface. In other words, Y is a compact connected nonorientable smooth manifold of dimension two, and Y has a covering by smooth coordinate charts such that each transition function is either holomorphic or anti-holomorphic. Any coordinate chart in the maximal atlas satisfying the above condition on transition functions will be called *compatible*. Such a nonorientable surface is called a *Klein surface*.

DEFINITION 2.1.

A *divisor* D on Y is a formal sum of type

$$D = \sum_{y\in Y} n_y y,$$

where $n_y \in \mathbb{Z}$ and $n_y = 0$ except for a finitely many points of Y.

DEFINITION 2.2.

The *degree* of a divisor $D = \sum_{y\in Y} n_y y$ is defined to be the integer $\deg(D) := \sum_{y\in Y} n_y$.

We will denote by $\mathrm{Div}(Y)$ the set of all divisors on Y. Let $\mathrm{Div}_d(Y) \subset \mathrm{Div}(Y)$ be the divisors of degree d.

Let $\pi : X \to Y$ be a double cover of Y given by local orientations on Y. So for a contractible open subset $U \subset Y$, the inverse image $\pi^{-1}(U)$ is two copies of U with the two possible orientations on U (see [1] for more details on Klein surfaces and their double covers).

Therefore, X is a Riemann surface, and the change of orientation defines an anti-holomorphic involution $\sigma : X \to X$ that commutes with π.

The involution σ induces in a natural way a mapping on the set of divisors on the Riemann surface X as follows

$$\begin{aligned} \sigma^* : \mathrm{Div}(X) &\longrightarrow \mathrm{Div}(X) \\ \sum m_j x_j &\longmapsto \sum m_j\, \sigma(x_j). \end{aligned}$$

Observe that σ^* preserves the degree.

Similarly, the quotient map $\pi : X \to Y$ induces mappings between the divisors on X and Y. To define those mappings we first set up some notation. For any point $y \in Y$ we will

denote by $\pi^{-1}(y)$ the divisor given by the inverse image of y. In other words, $\pi^{-1}(y) = x+\sigma(x)$, where $x \in X$ is a point satisfying $\pi(x) = y$. Then we can define two mappings as follows:

$$\pi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X), \qquad \pi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y),$$
$$\sum_{j=1}^{s} n_j y_j \mapsto \sum_{j=1}^{s} n_j \pi^{-1}(y_j), \qquad \sum_{j=1}^{s} m_j x_j \mapsto \sum_{j=1}^{s} m_j \pi(x_j).$$

Observe that $(\pi_* \circ \pi^*)(D) = 2D$ and $(\pi^* \circ \pi_*)(E) = E + \sigma^*(E)$ for $D \in \mathrm{Div}(Y)$ and $E \in \mathrm{Div}(X)$.

Let $\mathrm{Div}(X)^{\sigma^*}$ denote the set of fixed points of σ^* on $\mathrm{Div}(X)$.

The following lemma follows immediately from the above definitions.

Lemma 2.3. *The group* $\mathrm{Div}(Y)$ *is identified with* $\mathrm{Div}(X)^{\sigma^*}$. *The isomorphism takes the subgroup* $\mathrm{Div}_0(Y)$ *to* $\mathrm{Div}(X)_0^{\sigma^*} = \mathrm{Div}_0(X) \cap \mathrm{Div}(X)^{\sigma^*}$.

Let $j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the mapping induced by conjugation on the Riemann sphere $\widehat{\mathbb{C}}$, so that $j(z) = \overline{z}$ and $j(\infty) = \infty$. The quotient space is a surface with boundary, $\overline{\mathbb{H}} = \widehat{\mathbb{C}}/\langle j \rangle$. We can also identify $\overline{\mathbb{H}}$ with the closure of $\mathbb{H}$ (the upper half-plane) in the Riemann sphere. Let

$$p : \widehat{\mathbb{C}} \longrightarrow \overline{\mathbb{H}}$$

denote the quotient map. After identifying $\overline{\mathbb{H}}$ with the closure of $\mathbb{H}$ the map p coincides with the one defined by $p(x+\sqrt{-1}y) = x+\sqrt{-1}|y|$ and $p(\infty) = \infty$.

A *morphism* from Y to $\overline{\mathbb{H}}$ is a continuous mapping

$$u : Y \longrightarrow \overline{\mathbb{H}}$$

such that if (U, w) is a local coordinate function defined on Y, compatible with the Riemann surface structure, with $w(U) \subset \mathbb{H}$, then there exists a holomorphic function $F : w(U) \to \mathbb{C}$ that makes the following diagram commutative:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "w"'] & \overline{\mathbb{H}} \\ \mathbb{H} \arrow[r, "F"'] & \mathbb{C} \arrow[u, "p"'] \end{tikzcd},$$

where p is defined above.

Let u be a morphism, as above, from Y to $\overline{\mathbb{H}}$ which is not identically equal to 0 or ∞. If z_0 is a point of $\overline{\mathbb{H}}$, then by $u^{-1}(z_0)$ we understand the divisor given by the inverse image of z_0 under u (so the integers n_j in Definition 2.1 are given by the multiplicities of u at the corresponding points). Since 0 and ∞ in $\widehat{\mathbb{C}}$ project to two different points on $\overline{\mathbb{H}}$,

$$\mathrm{div}(u) := u^{-1}(0) - u^{-1}(\infty) \in \mathrm{Div}(Y)$$

is a divisor on Y.

DEFINITION 2.4.

A divisor $D \in \mathrm{Div}(Y)$ is called *principal* if $D = \mathrm{div}(u)$ for some morphism $u : Y \to \overline{\mathbb{H}}$ of the above type. The set of principal divisors of Y will be denoted by $\mathrm{Div}_P(Y)$.

PROPOSITION 2.5.

A divisor D on Y is principal if and only if there exists a divisor $E \in \mathrm{Div}_P(X) \cap \mathrm{Div}(X)^{\sigma^*}$ with $\pi^* D = E$.

Proof. Let $E = \mathrm{div}(f)$ be a principal divisor in $\mathrm{Div}(X)^{\sigma^*}$, where f is a non-constant meromorphic function on X. Consider the function ψ on X defined by $\psi(x) = \overline{f(\sigma(x))}$ on X. This function ψ is clearly meromorphic.

Since $E \in \mathrm{Div}(X)^{\sigma^*}$, we have $\mathrm{div}(\psi) = \mathrm{div}(f)$. Consequently, there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that $\psi = cf$.

Therefore, we have $f(x) = \psi(x)/c = \overline{f(\sigma(x))}/c = \overline{\psi(\sigma(x))}/|c|^2 = f(x)/|c|^2$. Take $c_0 \in \mathbb{C}$ with $c_0^2 = c$. Set $f_0 = c_0 f$.

The divisor for the meromorphic function f_0 coincides with E. Furthermore, f_0 satisfies the condition

$$f_0 \circ \sigma = \overline{f_0}\,.$$

Therefore, it induces a map

$$\hat{f} : Y := X/\sigma \longrightarrow \overline{\mathbb{H}} := \widehat{\mathbb{C}}/\langle j \rangle$$

with $\mathrm{div}(\hat{f}) = D$.

Conversely, let $D = \mathrm{div}(u)$ be a principal divisor on Y. Consider the composition $u \circ \pi : X \longrightarrow \overline{\mathbb{H}}$. It is straight-forward to see that the function $u \circ p$ lifts to a smooth function

$$f : X \longrightarrow \widehat{\mathbb{C}}$$

such that $p \circ f = u \circ \pi$. There are two such smooth lifts; one is holomorphic and the other is anti-holomorphic ($u \circ p$ also has a continuous lift, defined by the inclusion of $\overline{\mathbb{H}}$ in $\widehat{\mathbb{C}}$ which is not smooth). Let f denote the holomorphic one. Since $\mathrm{div}(f) = \pi^*(D) \in \mathrm{Div}(X)^{\sigma^*}$, the proof of the proposition is complete. □

DEFINITION 2.6.

The quotient of $\mathrm{Div}_0(Y)$, the group of all degree zero divisors on Y, by the subgroup of all principal divisors on Y is called the *Jacobian* of Y. The Jacobian of Y will be denoted by $J_0(Y)$.

From Proposition 2.5, it follows immediately that by sending any divisor D on Y to the divisor $\pi^* D$ on X we obtain an injective homomorphism from $J_0(Y)$ to the Jacobian $J_0(X)$ of X. From Lemma 2.3, it follows that $J_0(Y)$ coincides with the fixed point set of the involution of $J_0(X)$ defined by σ.

A function $f : W \to \mathbb{R}$, defined on an open subset of Y is called *harmonic* if for every point $y \in W$, there exists a compatible coordinate chart (U, w), with

$$y \in U \subseteq W\,,$$

such that the function $f \circ w^{-1}$ is harmonic. Since precomposition with holomorphic and anti-holomorphic functions preserve harmonicity, we conclude that harmonic functions are well-defined on Y.

We say that a real one-form η on Y is *harmonic* if it is locally given by $\mathrm{d}f$, where f is a harmonic function.

Let Ω denote the holomorphic cotangent bundle of the Riemann surface X. If $\omega \in H^0(X,\Omega)$ is given locally by $\omega = f\mathrm{d}z$, where f is a holomorphic function, then define

$$\overline{\sigma^*\omega} := (\overline{f\circ\sigma})\,\mathrm{d}(\overline{z}\circ\sigma)\,.$$

So if ω is defined over U, then $\overline{\sigma^*\omega}$ is a holomorphic one-form defined over $\sigma(U)$. More generally, for a one-form $\alpha = u\,\mathrm{d}z + v\,\mathrm{d}\overline{z}$, set

$$\sigma^*\alpha = (u\circ\sigma)\,\mathrm{d}(z\circ\sigma) + (v\circ\sigma)\,\mathrm{d}(\overline{z}\circ\sigma)\,.$$

Let $\mathscr{H}^1_{\mathbb{R}}(Y)$ and $\mathscr{H}^1_{\mathbb{R}}(X)$ denote the space of all real harmonic one-forms on Y and X respectively. Using the map $\pi : X \to Y$, we can lift harmonic forms on Y to smooth forms on X. It is easy to see that the pullback of a harmonic form on Y is a harmonic form on X. Therefore, there is a well-defined injective homomorphism $\pi^* : \mathscr{H}^1_{\mathbb{R}}(Y) \longrightarrow \mathscr{H}^1_{\mathbb{R}}(X)$.

The complex structure on X defines a Hodge-$*$ operator on one-forms on X. In local holomorphic coordinates the Hodge-$*$ operator is

$$*(u\,\mathrm{d}z + v\,\mathrm{d}\overline{z}) = -\sqrt{-1}u\,\mathrm{d}z + \sqrt{-1}v\mathrm{d}\overline{z}$$

or $*(a\,\mathrm{d}x + b\,\mathrm{d}y) = -b\,\mathrm{d}x + a\,\mathrm{d}y$.

A holomorphic one-form ω on X will be called σ-*invariant* if $\sigma^*\omega = \overline{\omega}$. The space of all σ-invariant forms on X will be denoted by $H^0(X,\Omega)^{\overline{\sigma^*}}$.

Theorem 2.7. *A holomorphic form $\omega \in H^0(X,\Omega)$ is σ-invariant if and only if there exists a form $\eta \in \mathscr{H}^1_{\mathbb{R}}(Y)$ such that $\omega = \beta + \sqrt{-1}(*\beta)$, where $\beta = \pi^*\eta$.*

The homomorphism $H^1_{\mathbb{R}}(Y) \longrightarrow H^0(X,\Omega)^{\overline{\sigma^*}}$ defined by

$$\eta \longmapsto \pi^*\eta + \sqrt{-1}(*\pi^*\eta)$$

is an isomorphism of real vector spaces.

Proof. Take any $\omega \in H^0(X,\Omega)$. Let $\omega = \beta + \sqrt{-1}(*\beta)$, where β is a real one-form. Now the condition $\sigma^*\omega = \overline{\omega}$ immediately implies that $\sigma^*\beta = \beta$. Therefore, β is the pullback of a form on Y. For any $\eta \in \mathscr{H}^1_{\mathbb{R}}(Y)$, the form $\pi^*\eta + \sqrt{-1}(*\pi^*\eta)$ is a σ-invariant holomorphic one-form.

Let

$$\varphi : \mathscr{H}^1_{\mathbb{R}}(Y) \longrightarrow H^0(X,\Omega)^{\overline{\sigma^*}}$$

be the homomorphism that sends any harmonic form $\eta \in \mathscr{H}^1_{\mathbb{R}}(Y)$ to the holomorphic form $\pi^*\eta + \sqrt{-1}(*\pi^*\eta)$. This homomorphism is injective since a holomorphic one-form with vanishing real part must be identically zero.

The inverse homomorphism

$$H^0(X,\Omega)^{\overline{\sigma^*}} \longrightarrow \mathscr{H}^1_{\mathbb{R}}(Y)$$

sends a σ-invariant form ω on Y to η with the property

$$\pi^*\eta = \frac{\omega + \overline{\omega}}{2}\,.$$

This completes the proof of the theorem. □

3. The Jacobian

A closed oriented smooth path γ on X gives an element $L_\gamma \in H^0(X,\Omega)^*$ defined by

$$L_\gamma(\omega) = \int_\gamma \omega,$$

where $\omega \in H^0(X,\Omega)$. Using Stokes' theorem we get a mapping from $H_1(X,\mathbb{Z})$ to $H^0(X,\Omega)^*$. The quotient space $H^0(X,\Omega)^*/H_1(X,\mathbb{Z})$ will be denoted by $J_1(X)$.

As we saw in the previous section, for a holomorphic one-form ω on X, the form $\overline{\sigma^*\omega}$ is again a holomorphic one-form. This involution of $H^0(X,\Omega)$ induces an involution

$$\sigma_1 : H^0(X,\Omega)^* \longrightarrow H^0(X,\Omega)^*.$$

In other words, $(\sigma_1(L))(\omega) = \overline{L(\overline{\sigma^*(\omega)})}$. It is easy to check that for any closed smooth-oriented path γ on X, the identity

$$\sigma_1(L_\gamma) = L_{\sigma(\gamma)}$$

is valid. So, the involution σ_1 preserves the subgroup $H_1(X,\mathbb{Z}) \subset H^0(X,\Omega)^*$.

Consequently, the involution σ_1 of $H^0(X,\Omega)^*$ induces an involution on the quotient space $J_1(X)$. The involution of $J_1(X)$ obtained this way will also be denoted by σ_1.

Let g be the genus of the compact connected Riemann surface X. Suppose we have a canonical basis of $H_1(X,\mathbb{Z})$, say $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$. This means that the corresponding intersection matrix is

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the identity matrix of rank g. Then there exists a unique basis of $H^0(X,\Omega)$, say $\{\omega_1,\dots,\omega_g\}$, such that $\int_{\alpha_k}\omega_j = \delta_{jk}$ ([2], Proposition III.2.8). We say that this basis is *adapted* to the given basis of homology.

Using this adapted basis we can identify $H^0(X,\Omega)^*$ with $\mathbb{C}^g$ by sending the element L of $H^0(X,\Omega)^*$ to the vector $(L(\omega_1),\dots,L(\omega_g))$.

Therefore, for any $\gamma \in H_1(X,\mathbb{Z})$, we may identify the element $L_\gamma \in H^0(X,\Omega)^*$ with

$$(L_\gamma(\omega_1),\dots,L_\gamma(\omega_g)) \in \mathbb{C}^g.$$

Denote by $\mathscr{L}$ the lattice in $\mathbb{C}^g$ defined by $H_1(X,\mathbb{Z})$ using this identification. The quotient space $J_1(X)$ defined earlier is clearly identified with the quotient $\mathbb{C}^g/\mathscr{L}$.

Assume that the basis $\{\omega_j\}$ is σ-invariant, that is, $\overline{\sigma^*(\omega_j)} = \omega_j$ for each $j \in [1,g]$. It is easy to check that by the above isomorphism of $H^0(X,\Omega)^*$ with $\mathbb{C}^g$ the involution σ_1 of $H^0(X,\Omega)^*$ (defined earlier) coincides with the conjugation defined as $(z_1,\dots,z_g) \longmapsto (\overline{z_1},\dots,\overline{z_g})$.

We will denote by $\sigma_\#$ the involution of $H_1(X,\mathbb{Z})$ induced by the involution σ of X. Let

$$\{\gamma_1,\dots,\gamma_g,\delta_1,\dots,\delta_g\}$$

be a canonical basis of $H_1(X,\mathbb{Z})$ satisfying the condition $\sigma_\#(\gamma_j) = \gamma_j$ for all $j \in [1,g]$. Let $\{\omega_1,\dots,\omega_g\}$ denote the corresponding adapted basis.

PROPOSITION 3.1.

The above adapted basis $\{\omega_1,\dots,\omega_g\}$ *is* σ*-invariant.*

Proof. Since

$$\int_{\sigma_\# \gamma} \omega = \int_\gamma \sigma^* \omega = \overline{\int_\gamma \sigma^* \overline{\omega}}$$

(as σ is an involution), the proposition follows immediately. $\square$

As in §2., let $J_0(X)$ denote the quotient $\mathrm{Div}_0(X)/\mathrm{Div}_P(X)$. For a meromorphic function f we have $\sigma^*(\mathrm{div}(f)) = \mathrm{div}(\overline{f \circ \sigma})$. So σ^* induces an involution on $J_0(X)$. This involution of $J_0(X)$ will be denoted by σ_0.

Let $\{\omega_1 \dots, \omega_g\}$ be the basis in Proposition 3.1. Recall the quotient $J_1(X)$ of $H^0(X,\Omega)^*$ defined earlier. The Abel–Jacobi map $A: X \to J_1(X)$ is defined as follows: choose a point x_0 of X and set $A(x) = \left[\int_{x_0}^x \omega_1,\dots,\int_{x_0}^x \omega_g\right]$, where the brackets denote the equivalence class in $J_1(X)$. We have

$$\begin{aligned} A(\sigma(x)) &= \left[\int_{x_0}^{\sigma(x)} \omega_1,\dots,\int_{x_0}^{\sigma(x)} \omega_g\right] = \left[\int_{x_0}^{\sigma(x_0)} \omega_1,\dots,\int_{x_0}^{\sigma(x_0)} \omega_g\right] \\ &\quad + \left[\int_{\sigma(x_0)}^{\sigma(x)} \omega_1,\dots,\int_{\sigma(x_0)}^{\sigma(x)} \omega_g\right] \\ &= c_0 + \left[\int_{\sigma(x_0)}^{\sigma(x)} \sigma^*(\omega_1),\dots,\int_{\sigma(x_0)}^{\sigma(x)} \sigma^*(\omega_1)\right] \\ &= c_0 + \left[\int_{x_0}^{x} \overline{\omega_1},\dots,\int_{x_0}^{x} \overline{\omega_g}\right] = c_0 + \overline{A(x)}, \end{aligned}$$

where $c_0 = A(\sigma(x_0))$. For a divisor $D = \sum_{j=1}^r n_j x_j$, we define

$$A(D) = \sum_{j=1}^{r} n_j A(x_j).$$

If D has degree equal to 0 then we can write it as $D = \sum_{j=1}^s x_j - \sum_{j=1}^s y_j$, where $x_j \neq y_k$ (though we can have repetitions among the x_js or the y_ks). Then it is easy to check that

$$A(\sigma_0(D)) = \overline{A(D)} = \sigma_1(A(D)), \tag{1}$$

where σ_1 and σ_0 are the earlier defined involutions of $H^0(X,\Omega)^*$ and $J_0(X)$ respectively.

By Abel's theorem, the map A can be extended to a map from $J_0(X)$ to $J_1(X)$. By the Abel–Jacobi inversion problem, the map $A: J_0(X) \to J_1(X)$ is surjective. Thus (1) says that σ_0 and σ_1 are equivalent under A, that is, the following diagram commutes:

$$\begin{tikzcd} J_0(X) \arrow[r, "A"] \arrow[d, "\sigma_0"'] & J_1(X) \arrow[d, "\sigma_1"] \\ J_0(X) \arrow[r, "A"'] & J_1(X). \end{tikzcd} \tag{2}$$

In the paragraph following Definition 2.6 we noted that the Jacobian $J_0(Y)$ coincides with the fixed point set of $J_0(X)$ for the action of the involution σ_0. Let $J_1(X)^{\sigma_1} \subset J_1(X)$ be the fixed point set for the action of the involution σ_1 on $J_1(X)$. From the commutativity of the diagram in (2) it follows immediately that $J_0(Y)$ is identified with $J_1(X)^{\sigma_1}$. Finally using Theorem 2.7, the Jacobian $J_0(Y)$ is identified with the quotient of $\mathscr{H}^1_{\mathbb{R}}(Y)$ by the torsion-free part of $H_1(Y,\mathbb{Z})$.

4. Line bundles on a Klein surface

Let L be a holomorphic line bundle over a Riemann surface X. By $\overline{L}$ we will mean the C^∞ complex line bundle over X whose transition functions are the conjugations of the transition functions for L. To explain this, let U_i, $i \in I$, be an open covering of X and assume that over each U_i we are given a holomorphic trivialization of L. So for any ordered pair $i, j \in I$, we have the corresponding transition function

$$f_{i,j} : U_i \cap U_j \longrightarrow \mathbb{C}^*$$

which is holomorphic. The C^∞ complex line bundle $\overline{L}$ has C^∞ trivializations over each U_i, $i \in I$, and for any ordered pair $i, j \in I$ the corresponding transition function is $\overline{f_{i,j}}$. It is easy to see that the collection $\{\overline{f_{i,j}}\}_{i,j\in I}$ satisfy the cocycle condition to define a C^∞ complex line bundle.

The line bundle $\overline{L}$ can also be defined without using local trivializations. A C^∞ complex line bundle is a C^∞ real vector bundle of rank two together with a smoothly varying complex structure on the fibers (which are real vector spaces of dimension two). The underlying real vector bundle of rank two for $\overline{L}$ coincides with the one for L. For any $x \in X$, if J_x is the complex structure on the fiber L_x, then the complex structure of the fiber $\overline{L}_x$ is $-J_x$.

As in §2., let Y be a nonorientable Klein surface and X its double cover, which is a connected Riemann surface of genus g.

Let L be a holomorphic line bundle over X. The complex line bundle $\sigma^*\overline{L}$ has a natural holomorphic structure, where σ, as before, is the involution of X. To construct the holomorphic structure on $\sigma^*\overline{L}$, observe that if f is a holomorphic function on an open subset U of X, then $\overline{f\circ\sigma}$ is a holomorphic function of $\sigma(U)$. We can choose the above open subsets U_i (sets over which L is trivialized) in such a way that $\sigma(U_i) = U_i$. Now, since each $\overline{f_{i,j}\circ\sigma}$ is a holomorphic function on $U_i \cap U_j$, the complex line bundle $\sigma^*\overline{L}$ gets equipped with a holomorphic structure.

PROPOSITION 4.1.

Let D be a divisor on X of degree d and L the corresponding holomorphic line bundle $\mathscr{O}_X(D)$ over X of degree d. Then the holomorphic line bundle $\sigma^*\overline{L}$ corresponds to the divisor $\sigma(D)$, that is, $\sigma^*\overline{L} \cong \mathscr{O}_X(\sigma(D))$.

Proof. Since $L \cong \mathscr{O}_X(D)$, we have a meromorphic section s of L with the positive part of D as the zeros of s (of order given by multiplicity) and the negative part of D as the poles of s (of order given by multiplicity). Since L and $\overline{L}$ are identified as real rank two vector bundles, the pullback σ^*s defines a smooth section of $\sigma^*\overline{L}$ over the complement (in X) of the support of D.

It is straight-forward to check that the section $\sigma^* s$ of $\sigma^*\overline{L}$ is meromorphic. The divisor defined by the meromorphic section $\sigma^* s$ clearly coincides with $\sigma(D)$. Consequently, $\sigma^*\overline{L}$ is holomorphically isomorphic to the line bundle over X defined by the divisor $\sigma(D)$. This completes the proof of the proposition. $\square$

Recall the quotient space $J_0(X) := \mathrm{Div}_0(X)/\mathrm{Div}_P(X)$ considered in §2.. The Jacobian $J_0(X)$ is identified with the space of all isomorphism classes of degree zero holomorphic line bundles over X. The isomorphism sends any divisor D to the line bundle $\mathscr{O}_X(D)$. As in §3., let σ_0 denote the involution of $J_0(X)$ defined by σ. From Proposition 4.1, it follows immediately that the above identification of $J_0(X)$ with degree zero line bundles takes the involution σ_0 to the involution defined by $L \longmapsto \sigma^*\overline{L}$ on the space of all isomorphism classes of degree zero line bundles.

Let D be a divisor of degree zero on the nonorientable Klein surface Y. From Proposition 2.5, it follows immediately that D is principal if and only if $\pi^* D$ is principal. Therefore, we have an injective homomorphism

$$\rho : \frac{\mathrm{Div}_0(Y)}{\mathrm{Div}_P(Y)} \longrightarrow \frac{\mathrm{Div}_0(X)}{\mathrm{Div}_P(X)} = J_0(X) \tag{3}$$

defined by $D \longmapsto \pi^* D$, where $\mathrm{Div}_P(Y)$ denotes the group of principal divisors on Y (as before, Div_0 denotes degree zero divisors).

Theorem 4.2. *The image of the homomorphism ρ in* (3) *coincides with the subgroup of $J_0(X)$ defined by all holomorphic line bundle L with $\sigma^*\overline{L}$ holomorphically isomorphic to L.*

Proof. Let D be a divisor on Y of degree zero. The divisor $\pi^* D$ on X is left invariant by the action of the involution σ. From the above remark that the involution σ_0 is taken into the involution defined by $L \longmapsto \sigma^*\overline{L}$, it follows immediately that the holomorphic line bundle $L = \mathscr{O}_X(\pi^* D)$ over X corresponding to the divisor $\pi^* D$ satisfies the condition $L \cong \sigma^*\overline{L}$.

For the converse direction, take a holomorphic line bundle L over X which has the property that $\sigma^*\overline{L}$ is isomorphic to L. Let s be a nonzero meromorphic section of L. If the divisor $\mathrm{div}(s)$ is left invariant by the involution σ, then L is in the image of ρ.

If $\mathrm{div}(s)$ is *not* left invariant by the involution σ, then consider the meromorphic section of $\sigma^*\overline{L}$ defined by $\sigma^* s$. (Recall that $\sigma^*\overline{L}$ and $\sigma^* L$ are identified as real rank two C^∞ bundles, and the section of $\sigma^*\overline{L}$ defined by $\sigma^* s$ using this identification is meromorphic.)

Now, fix a holomorphic isomorphism

$$\alpha : L \longrightarrow \sigma^*\overline{L} \tag{4}$$

such that the composition

$$L \xrightarrow{\alpha} \sigma^*\overline{L} \xrightarrow{\sigma^*\overline{\alpha}} \overline{\sigma^*\overline{\sigma^*\overline{L}}} = L \tag{5}$$

is the identity automorphism of L, where $\overline{\alpha}$ is the isomorphism of $\overline{L}$ with $\overline{\sigma^*\overline{L}}$ induced by α. Note that such an isomorphism exists. Indeed, if

$$\alpha' : L \longrightarrow \sigma^*\overline{L}$$

is any isomorphism, then the automorphism $\sigma^*\overline{\alpha'} \circ \alpha'$ of L (defined as in (5)) is the multiplication by a nonzero scalar $c \in \mathbb{C}$. Take any $c_0 \in \mathbb{C}$ such that $c_0^2 = c$. Now the isomorphism $\alpha = \alpha'/c_0$ satisfies the condition that the composition in (5) is the identity automorphism of L.

Let s' be the meromorphic section of L defined by the above section $\sigma^* s$ using this isomorphism. Consider the meromorphic section $s' + s$ of L. Since $\mathrm{div}(s)$ is not left invariant by σ, this meromorphic section $s' + s$ is not identically zero. The divisor $\mathrm{div}(s + s')$ is clearly left invariant by the involution σ. Hence $L \in J_0(X)$ is in the image of ρ. This completes the proof of the theorem. $\square$

5. Nonorientable line bundle

In this section we will define a line bundle on Y intrinsically without using X.

Let $\{U_i\}_{i \in I}$ be a covering of Y by open subsets and for each U_i,

$$\phi_i : U_i \longrightarrow \mathbb{R}^2,$$

a C^∞ coordinate chart. Consider the trivial (real) line bundle $U_i \times \mathbb{R}$ on each U_i. Using

$$\frac{\det \mathrm{d}(\phi_j \circ \phi_i^{-1})}{|\det \mathrm{d}(\phi_j \circ \phi_i^{-1})|} \in \pm 1 \subset \mathrm{Aut}(\mathbb{R})$$

as the transition function over $U_i \cap U_j$ for the pair (i, j), we get a real line bundle over Y. This line bundle will be denoted by ξ. Since the transition functions are ± 1, the line bundle $\xi^{\otimes 2}$ has a natural isomorphism with the trivial line bundle $Y \times \mathbb{R}$. Let

$$\lambda : \xi^{\otimes 2} \longrightarrow Y \times \mathbb{R} \tag{6}$$

be the isomorphism.

We will give a construction of the line bundle ξ without using coordinate charts. Consider the complement $\bigwedge^2 TY \setminus \{0_Y\}$ of the zero section of the real line bundle $\bigwedge^2 TY$, where TY is the real tangent bundle of Y. The multiplicative group

$$\mathbb{R}^+ := \{c \in \mathbb{R} \,|\, c > 0\}$$

acts on $\bigwedge^2 TY \setminus \{0\}$. The action of any $c \in \mathbb{R}^*$ sends any $v \in \bigwedge^2 TY \setminus \{0\}$ to cv. Also, the multiplicative group ± 1 acts on $\bigwedge^2 TY \setminus \{0\}$ by sending any v to $\pm v$. Since these two actions commute, we have an action of the multiplicative group ± 1 on

$$Z := \frac{\bigwedge^2 TY \setminus \{0_Y\}}{\mathbb{R}^+}.$$

Now, we have

$$\xi = \frac{Z \times \mathbb{R}}{\pm 1},$$

where ± 1 acts diagonally and it acts on $\mathbb{R}$ as multiplication by ± 1.

We will show that the Klein surface (nonorientable complex) structure on Y gives an isomorphism of TY with $TY \otimes \xi$, where TY as before is the (real) tangent bundle of Y. To construct the isomorphism, take a compatible coordinate chart

$$\phi_i : U_i \longrightarrow \mathbb{C}$$

compatible with the nonorientable complex structure. The orientation of the complex line $\mathbb{C}$ induces an orientation of U_i using ϕ_i. This gives a trivialization of ξ over U_i (this

induced trivialization is also clear from the first construction of ξ). Using ϕ_i we have a complex structure on U_i obtained from the complex structure of $\mathbb{C}$. Let

$$\gamma_i : TU_i \longrightarrow TU_i \otimes \xi|_{U_i}$$

be the isomorphism defined by the almost complex structure of U_i and the trivialization of $\xi|_{U_i}$. If ϕ_j is another compatible coordinate chart then the function $\phi_i \circ \phi_j^{-1}$ is either holomorphic or anti-holomorphic. This immediately implies that the isomorphism

$$\gamma_j : TU_j \longrightarrow TU_j \otimes \xi|_{U_j}$$

(obtained by repeating the construction of γ_i for the new compatible coordinate chart) coincides with γ_i over $U_i \cap U_j$. Consequently, the locally defined isomorphisms $\{\gamma_i\}$ patch together compatibly to give a global isomorphism

$$\gamma : TY \longrightarrow TY \otimes \xi \tag{7}$$

over Y.

A *nonorientable complex line bundle* over Y is a C^∞ real vector bundle of rank two over Y together with a C^∞ isomorphism of vector bundles

$$\tau : E \longrightarrow E \otimes \xi \tag{8}$$

satisfying the condition that the composition

$$E \xrightarrow{\tau} E \otimes \xi \xrightarrow{\tau \otimes \mathrm{Id}_\xi} (E \otimes \xi) \otimes \xi = E \otimes \xi^{\otimes 2} \xrightarrow{\mathrm{Id}_E \otimes \lambda} E \tag{9}$$

coincides with the automorphism of E defined by multiplication with -1, where λ is defined in (6).

Therefore, if for a point $y \in Y$ we fix $w \in \xi_y$ with $\lambda(w \otimes w) = 1$, then the automorphism of the fiber E_y defined by

$$v \longmapsto \langle \tau(v), w^* \rangle$$

is an almost complex structure on E_y, where $\langle -, - \rangle$ denotes the contraction of ξ_y with its dual line ξ_y^* and $w^* \in \xi_y^*$ is the dual element of w, that is, $\langle w, w^* \rangle = 1$.

Let (E, τ) be a nonorientable complex line bundle over Y as above. It is easy to see that the C^∞ vector bundle E is *not* orientable. Indeed, the two orientations on a two-dimensional real vector space V defined by J and $-J$, where J is an almost complex structure on V, are opposite to each other. To explain this, note that an orientation of the tangent space T_yY, where $y \in Y$, induces an orientation of the fiber E_y and conversely. Indeed, giving an orientation of T_yY is equivalent to giving a vector in $w \in \xi_y$ with $\lambda(w \otimes w) = 1$. As it was shown above, such an element w gives an almost complex structure on E_y. Hence E_y gets an orientation. Conversely, if we have an orientation of the fiber E_y, then choose the element $w \in \xi_y$, with $\lambda(w \otimes w) = 1$, that induces this orientation using τ. Now, w gives an orientation of T_yY. Therefore, giving an orientation of E_y is equivalent to giving an orientation of T_yY. Since the tangent bundle TY is not orientable, we conclude that the vector bundle E is not orientable.

The total space of the vector bundle E will also be denoted by E. Let

$$f : E \longrightarrow Y$$

be the natural projection. Note that the relative tangent bundle for f (that is, the kernel of the differential $\mathrm{d}f$) is identified with f^*E. So we have the following exact sequence of vector bundle

$$0 \longrightarrow f^*E \longrightarrow TE \longrightarrow f^*TY \longrightarrow 0 \tag{10}$$

over the manifold E.

The line bundle $f^*\xi$ will be denoted by $\hat{\xi}$. Let

$$J : TE \longrightarrow TE \otimes \hat{\xi}$$

be an isomorphism such that the composition

$$TE \xrightarrow{J} TE \otimes \hat{\xi} \xrightarrow{J\otimes \mathrm{Id}_{\hat{\xi}}} (TE \otimes \hat{\xi}) \otimes \hat{\xi} = TE \otimes \hat{\xi}^{\otimes 2} \xrightarrow{\mathrm{Id}_E \otimes f^*\lambda} TE$$

coincides with the automorphism of E defined by multiplication with -1. Assume that the isomorphism J satisfies the following further conditions:

(1) The subbundle f^*E in (10) is preserved by J and $J|_{f^*E}$ coincides with the isomorphism $f^*\tau$, where τ is defined in (8).
(2) The action of J on the quotient f^*TY in (10) coincides with the isomorphism $f^*\gamma$, where γ is constructed in (7).

A holomorphic structure on the nonorientable complex line bundle E is an isomorphism J as above satisfying the following conditions (apart from the above conditions) described below.

If we take a coordinate chart (U,ϕ) on Y compatible with the nonorientable Riemann surface structure, then as we saw before, the restriction $\xi|_U$ gets a trivialization. This in turn gives a trivialization of $\hat{\xi}$ over $f^{-1}(U)$. Using this trivialization of $\hat{\xi}|_{f^{-1}(U)}$, the isomorphism $J|_{f^{-1}(U)}$ becomes an automorphism J_ϕ of $(TE)_{f^{-1}(U)}$ with the property that $J_\phi \circ J_\phi$ coincides with the automorphism of $(TE)_{f^{-1}(U)}$ given by multiplication with -1. In other words, J_ϕ is an almost complex structure on $f^{-1}(U)$.

A *holomorphic structure* on the nonorientable complex line bundle E is an isomorphism J satisfying the following two conditions (apart from the earlier conditions):

(1) The almost complex structure J_ϕ on $f^{-1}(U)$ is integrable for every compatible coordinate chart.
(2) There is a homomorphic isomorphism

$$f_\phi : f^{-1}(U) \longrightarrow \phi(U) \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$$

that fits in a commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f_\phi} & \phi(U) \times \mathbb{C} \\ \Big\downarrow f & & \Big\downarrow \\ U & \xrightarrow{\phi} & \phi(U) \end{array}$$

(the right vertical arrow is the projection to the first coordinate), and the restriction of f_ϕ to any fiber of f is a complex linear isomorphism with $\mathbb{C}$.

A *holomorphic line bundle* over Y is defined to be a complex line bundle equipped with a holomorphic structure.

As in §2., let $\pi : X \longrightarrow Y$ be the double cover of the nonorientable Riemann surface Y given by local orientations. As before, let σ denote the anti-holomorphic involution of the Riemann surface X.

Theorem 5.1. *The space of all holomorphic line bundles over Y are in bijective correspondence with the holomorphic line bundles L over X with the property that $\sigma^*\overline{L}$ is holomorphically isomorphic to L.*

Proof. Let L be a holomorphic line bundle over X such that $\sigma^*\overline{L}$ is holomorphically isomorphic to L. Fix an isomorphism

$$\alpha : L \longrightarrow \sigma^*\overline{L}$$

as in (4) such that the composition in (5) is the identity automorphism of L.

Since the underlying C^∞ line bundle for $\overline{L}$ is identified with that of L, the isomorphism α gives a C^∞ isomorphism of L with σ^*L whose composition with itself is the identity automorphism of L. In other words, α is a C^∞ lift to L of the involution σ of X. Therefore, the quotient L/α is a real vector bundle of rank two over $X/\sigma = Y$. This real vector bundle of rank two over Y will be denoted by E.

To construct a complex structure on E, first note that the (real) line bundle $\pi^*\xi$ over X is canonically trivialized, i.e., there is a natural isomorphism of $\pi^*\xi$ with the trivial line bundle $X \times \mathbb{R}$ over X. Indeed, this follows immediately from the definitions of X and ξ. The complex structure on the fibers on L give an isomorphism

$$L \longrightarrow L$$

defined by multiplication by $\sqrt{-1}$. Consider the composition

$$L \longrightarrow L \longrightarrow L \otimes_{\mathbb{R}} (X \times \mathbb{R}) \longrightarrow L \otimes_{\mathbb{R}} \pi^*\xi$$

which we denote by J_0. Since $\pi^*\xi$ is the pullback of a line bundle over Y, there is a natural lift of the involution σ to $\pi^*\xi$. On the other hand, α is a C^∞ lift of the involution σ to L. Therefore, we have a lift of the involution σ to $L \otimes_{\mathbb{R}} \pi^*\xi$. It is straight-forward to check that the isomorphism J_0 defined above commutes with the lifts of the involution σ to L and $L \otimes_{\mathbb{R}} \pi^*\xi$. This immediately implies that the isomorphism J_0 descends to an isomorphism of E with $E \otimes_{\mathbb{R}} \xi$ over Y. This isomorphism of E with $E \otimes_{\mathbb{R}} \xi$, which we denote by J, clearly satisfies the condition that the composition in (9) is multiplication by -1. Therefore, (E, J) is a nonorientable complex line bundle.

It is easy to see that J defines a holomorphic structure on E. Indeed, this is an immediate consequence of the fact that the almost complex structure on the total space of L is integrable.

For the converse direction, take a holomorphic line bundle (E, J) over Y. Consider the (real) rank two C^∞ vector bundle π^*E over X. Since $\pi^*\xi$ is identified with the trivial line bundle, the complex structure τ on E (defined in (8)) gives a complex structure on π^*E. For the same reason, J defines an integrable complex structure on the total space of π^*E. Using the conditions on J the vector bundle π^*E gets the structure of a holomorphic line bundle over X.

Since π^*E is the pullback of a vector bundle over Y, the involution σ of X has a natural C^∞ lift to π^*E. The isomorphism of π^*E with $\sigma^*\pi^*E$ defined by this lift gives a holomorphic isomorphism of the holomorphic line bundle π^*E with $\sigma^*\overline{\pi^*E}$. This completes the proof of the theorem.

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